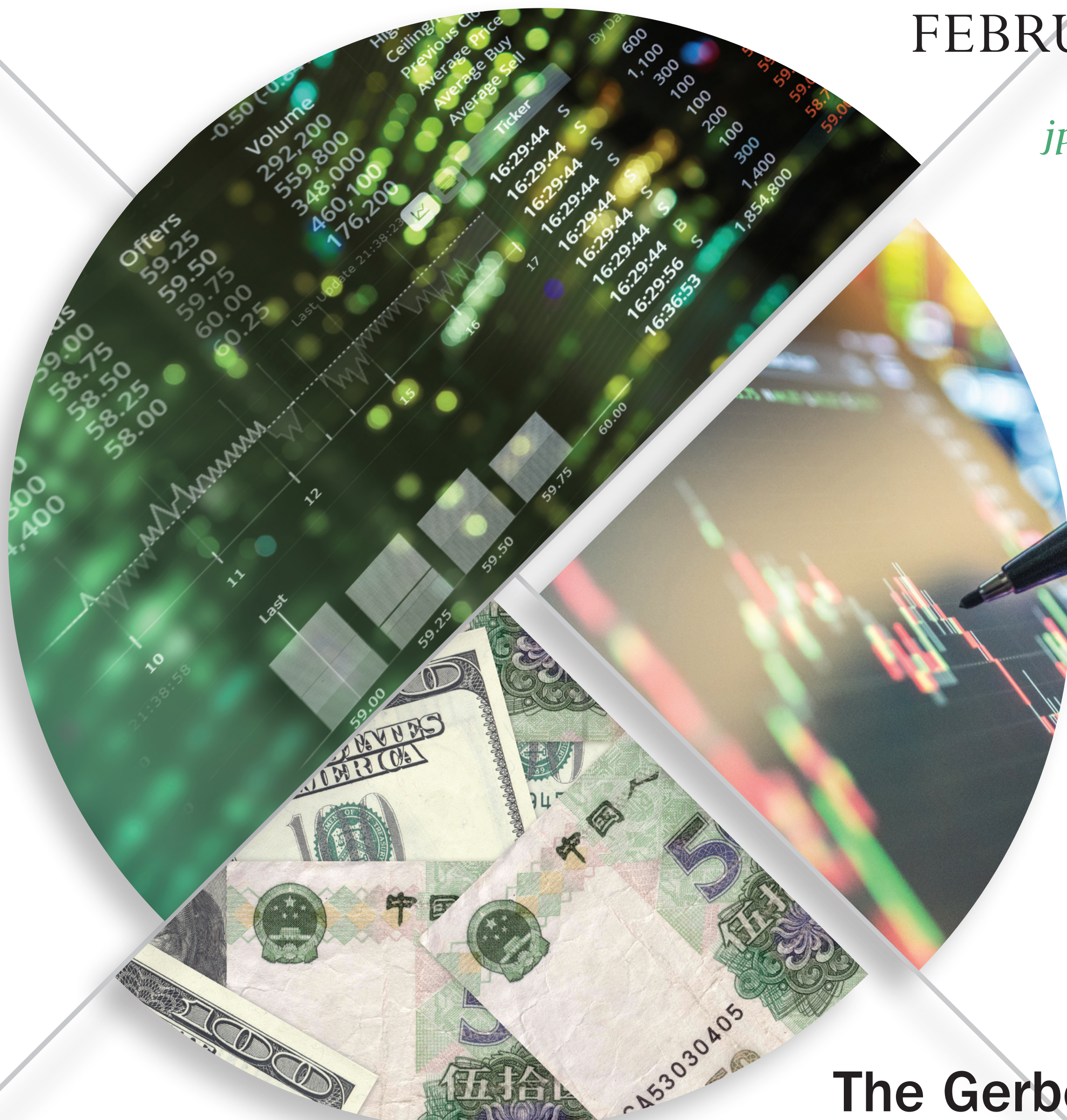
PORTFOLIO MANAGEMENT RESEARCH
By With Intelligence

the journal of PORTFOLIO management

volume 48 number 3

FEBRUARY 2022

jpm.pm-research.com



The Gerber Statistic: A Robust Co-Movement Measure for Portfolio Optimization

Sander Gerber, Harry M. Markowitz, Philip A. Ernst, Yinsen Miao, Babak Javid, and Paul Sargen

HUDSON
BAY
CAPITAL

Hudson Bay Capital Management LP is a manager of alternative investment opportunities in the global markets targeting traditional and non-traditional sources of alpha by employing a diverse set of catalyst-driven absolute return strategies that are intended to be uncorrelated to each other and to the major indices. Our multiple portfolio teams invest across the corporate capital structure, often in conjunction with an event or catalyst, in an effort to exploit market inefficiencies. Founded in December 2005, with Sander Gerber as Chief Investment Officer, the firm seeks to achieve persistently positive returns while maintaining a focus on risk management and capital preservation.

Sander Gerber

Sander Gerber is Chief Executive Officer and Chief Investment Officer of Hudson Bay Capital, a multi-strategy hedge fund manager investing globally. Mr. Gerber has more than 30 years of investing experience in multiple securities classes and derivatives across a broad range of strategies.

In 2008, Mr. Gerber developed the Gerber Statistic, which was accepted as an innovation complementary to his own work by Dr. Harry Markowitz, the Nobel Prize-winning economist and father of Modern Portfolio Theory (MPT). The Gerber Statistic is utilized by Hudson Bay to identify the co-movement of financial assets, enabling early detection of concentration risks and insufficient diversification. Subsequently, Messrs. Gerber and Markowitz, in a landmark research paper published in *The Journal of Portfolio Management,* gave further validation to replacing historical correlation in the calculation of covariance with the Gerber Statistic. The result, as described in *Institutional Investor,* "leads to outperformance in cumulative return, average geometric return, and Sharpe ratio."

Mr. Gerber began his investment career in 1991, as a member of the American Stock Exchange working as an equity options market maker. In 1997, he founded Gerber Asset Management to develop and engage in proprietary investment strategies. In late 2005, Mr. Gerber and Yoav Roth co-founded Hudson Bay Capital, which concentrates on generating positive returns while maintaining a focus on risk management and capital preservation.

Mr. Gerber is a member of the Council on Foreign Relations and the Economic Club of New York. He is a fellow and board member of the Jerusalem Center for Public Affairs and serves as a member of the United States Agency for International Development's Partnership for Peace Fund Advisory Board. Mr. Gerber served as a member of the Senior Advisory Group to the Director of National Intelligence from 2017–2019. Formerly, Mr. Gerber was the Vice Chairman of the Woodrow Wilson International Center for Scholars, and Chairman of its Investment Committee.

Mr. Gerber graduated cum laude from the University of Pennsylvania, with a BSE in Finance from Wharton and a BA in Humanistic Philosophy from the College of Arts and Sciences.

Harry M. Markowitz

Recipient of the Nobel Memorial Prize in Economic Sciences

Dr. Harry Markowitz is a Nobel Prize winning economist who devised the modern portfolio theory, introduced to academic circles in his article, "Portfolio Selection," which appeared in the Journal of Finance in 1952. Dr. Markowitz's theories emphasized the importance of portfolios, risk, the correlations between securities, and diversification. His work, in collaboration with Merton H. Miller and William F. Sharpe, changed the way that people invested. These three intellectuals shared the 1990 Nobel Prize in Economics.

After taking an early interest in physics and philosophy, Dr. Markowitz received a B.A. from the University of Chicago. It was here that he decided to continue his studies focusing on economics. During this time, he had the opportunity to study under important economists including Milton Friedman, Tjalling Koopmans, Jacob Marschak and Leonard Savage. While still a student, he was invited to become a member of the Cowles Commission.

While writing his dissertation Dr. Markowitz chose to apply mathematics to the analysis of the stock market. This led to the development of his seminal theory of portfolio allocation under uncertainty, published in 1952 by the Journal of Finance. In the same year, Dr. Markowitz went to work for the RAND Corporation, where he met George Dantzig. With Dantzig's help, Dr. Markowitz started to research optimization techniques, developing the critical line algorithm for the identifications of the optimal mean-variance portfolios, relying on what was later named the Markowitz Frontier.

In 1955, he received a Ph.D. from the University of Chicago with a thesis on the portfolio theory. The topic was so novel that while Markowitz was defending his dissertation, Milton Friedman jokingly argued that portfolio theory was not economics. During 1955-1956 Markowitz spent a year at the Cowles Foundation, which had moved to Yale University, at the invitation of James Tobin. He published the critical line algorithm in a 1956 paper and used the time at the foundation to write a book on portfolio allocation which was published in 1959.

Markowitz won the Nobel Prize in Economics in 1990, while a professor of finance at Baruch College of the City University of New York.

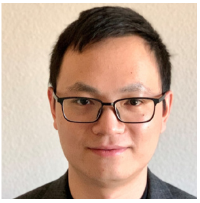
Philip A. Ernst

Philip Ernst is an Associate Professor with tenure at Rice University. His research interests include exact distribution theory, mathematical finance, operations research, optimal stopping, queueing systems, statistical inference for stochastic processes, and stochastic control. He is an associate editor for Mathematics of Operations Research, an associate editor for Stochastics, an associate editor for Statistics and Probability Letters, and an associate editor of Journal of Stochastic Analysis. He is also Guest Editor-in-Chief of "In Memoriam: Larry Shepp," a special issue of Stochastic Processes and their Applications to appear in Spring 2022.

Ernst's research is funded by the U.S. Office of Naval Research (ONR), U.S. Army Research Office (ARO), and the National Science Foundation (NSF). He is the recipient of numerous international and national research awards. In 2020, Mr. Ernst received the (inaugural) INFORMS Donald P. Gaver, Jr. Early Career Award for Excellence in Operations Research. In 2018, Mr. Ernst was honored with the Tweedie New Researcher Award from the Institute of Mathematical Statistics, widely considered the highest honor for excellence in research for an early-career mathematical statistician or applied probabilist. In that same year, Mr. Ernst also received the prestigious Army Research (ARO) Young Investigator Award.

Mr. Ernst is also strongly committed to teaching and mentoring. In 2021, he was the recipient of the George R. Brown Award for Superior Teaching at Rice University. In 2017, Mr. Ernst was the recipient of the Nicolas Salgo Distinguished Teacher Award, the oldest teaching award given to faculty members at Rice University. In 2016, Mr. Ernst was the recipient of the Sophia Meyer Farb Prize for Teaching, given annually to one Rice assistant professor for excellence in teaching. In 2015, he received the Graduate Student Association Mentoring Award, given annually to two faculty members.

Mr. Ernst earned his Ph.D. in statistics from the Wharton School of the University of Pennsylvania in 2014.



Yinsen Miao

Yinsen Miao is a senior data scientist from Fidelity Investments. Mr. Miao's research interests include portfolio selection, statistical machine learning, and Bayesian hierarchical modeling.

Mr. Miao received his Ph.D. in Statistics from Rice University.

Babak Javid

Babak Javid is a senior risk analyst at Hudson Bay Capital. Prior to joining Hudson Bay Capital in 2014, Mr. Javid worked with various startup companies in the Bay Area, California, one of which went public in 2017. Mr. Javid was the silver medal winner of the National Mathematics Olympiad in Iran in both 1998 and 1999.

Mr. Javid received his M.S. in Financial Mathematics from Stanford University, Stanford, CA in 2010, his M.Sc. in electrical engineering from the University of Toronto, Canada in 2006 and his B.Sc. in electrical engineering from Sharif University of Technology, Iran in 2004.

Paul Sargen

Paul Sargen is a Partner and the Chief Risk Officer at Hudson Bay Capital Management LP responsible for supporting Sander Gerber, Chief Investment Officer, in overseeing the risk management function across the portfolio. Prior to joining Hudson Bay, Mr. Sargen was a Director in the Quantitative Trading Group at Aristeia Capital, LLC, a convertible arbitrage, distressed debt, special situation/event-driven hedge fund manager. At Aristeia, Mr. Sargen was involved in all aspects of risk management, from risk identification and systems development to hedge selection and trading. His experience includes valuing and evaluating a variety of investment products covering most major asset classes, such as convertibles, bank capital, equity options, corporate bonds and preferreds, sovereign debt, CDS, interest rate swaps and futures, and FX forwards and futures.

Mr. Sargen received his BA in Economics with a Minor in Mathematics from Stanford University.

The Gerber Statistic: A Robust Co-Movement Measure for Portfolio Optimization

Sander Gerber, Harry M. Markowitz, Philip A. Ernst, Yinsen Miao, Babak Javid, and Paul Sargen

Sander Gerber
is chief executive officer at Hudson Bay Capital Management in New York, NY.
sgerber@hudsonbaycapital.com

Harry M. Markowitz
is an adjunct professor of finance and accounting in the Rady School of Management at University of California in La Jolla, CA.
hmm@aol.com

Philip A. Ernst
is an associate professor of statistics at Rice University in Houston, TX.
pe6@rice.edu

Yinsen Miao
is a senior data scientist at Fidelity Investments in Boston, MA.
yinsen.miao@fmr.com

Babak Javid
is a senior analyst at Hudson Bay Capital Management in New York, NY.
bjavid@hudsonbaycapital.com

Paul Sargen
is chief risk officer at Hudson Bay Capital Management in New York, NY.
psargen@hudsonbaycapital.com

KEY FINDINGS

- We introduce the Gerber statistic, a robust co-movement measure for covariance matrix estimation for the purpose of portfolio construction.
- The Gerber statistic is designed to recognize co-movement between series when the movements are substantial and to be insensitive to small co-movements that may be due to noise alone.
- Using a well-diversified portfolio of nine assets over a 30-year period (January 1990–December 2020), we empirically find that, for almost all investment scenarios considered, the Gerber statistic's returns dominate those achieved by both historical covariance and by the shrinkage method of Ledoit and Wolf.

ABSTRACT

The purpose of this article is to introduce the Gerber statistic, a robust co-movement measure for covariance matrix estimation for the purpose of portfolio construction. The Gerber statistic extends Kendall's Tau by counting the proportion of simultaneous co-movements in series when their amplitudes exceed data-dependent thresholds. Because the statistic is not affected by extremely large or extremely small movements, it is especially well suited for financial time series, which often exhibit extreme movements and a great amount of noise. Operating within the mean–variance portfolio optimization framework of Markowitz, we consider the performance of the Gerber statistic against two other commonly used methods for estimating the covariance matrix of stock returns: the sample covariance matrix (also called the *historical covariance matrix*) and shrinkage of the sample covariance matrix given by Ledoit and Wolf. Using a well-diversified portfolio of nine assets over a 30-year period (January 1990–December 2020), we find, empirically, that for almost all investment scenarios considered, the Gerber statistic's returns dominate those achieved by both historical covariance and by the shrinkage method of Ledoit and Wolf.

Portfolio construction (Markowitz 1952, 1959) relies heavily on the availability of the matrix of covariances between securities' returns. Often the sample covariance matrix is used as an estimate for the actual covariance matrix (Jobson and Korkie 1980). As early as Sharpe (1963), however, various models have been used to ease the computational burden and to improve the statistical properties of covariance matrix estimates. Nevertheless, a central problem still exists with many covariance matrix estimators: they employ product-moment–based estimates that are inherently not robust. This is particularly troublesome if the underlying distribution of returns contains extreme measurements or outliers. Robust estimators, based on the pioneering work of Tukey (1960), Hampel (1968, 1974), and Huber (1977), have largely overcome this problem. Shevlyakov and Smirnov (2011) provide a thorough examination of modern robust methods for computing correlations.

However, financial time series have characteristics that make even standard robust techniques unsuitable. Financial time series are particularly noisy, and this noise can be easily misinterpreted as information. One consequence, for example, is that correlation matrix estimates (even those constructed with robust techniques) often have non-zero entries corresponding to series that in fact have no meaningful correlation. On the other hand, correlation estimates can also be distorted if the series contains extremely large (either positive or negative) observations.

The key purpose of this article is to introduce the Gerber statistic (GS),[1] a robust co-movement measure that ignores fluctuations below a certain threshold and simultaneously limits the effects of extreme movements. The GS is designed to recognize co-movement between series when the movements are substantial and to be insensitive to small co-movements that may be due to noise alone. The GS is similar to Kendall's Tau (Kendall 1938) in that it also measures co-movement between two groups of data as a function of the difference between the number of concordant and discordant pairs (see the following section and the Appendix). However, the GS generalizes Kendall's Tau because it includes thresholds such that only co-movements that exceed thresholds are recognized as being either concordant or discordant.

In the present article, we confine our analysis to the mean–variance optimization (MVO) framework of Markowitz (1952, 1959). Within the mean–variance paradigm, we compare the performance of the GS with two commonly employed covariance matrix estimators of stock returns: (1) the sample covariance matrix (also referred to as the historical covariance (HC) matrix, or simply *historical covariance*) and (2) the shrinkage estimator of Ledoit and Wolf (2004), which shrinks the sample covariance matrix toward a structural estimator. It is well known that the sample covariance matrix lacks robustness and is highly susceptible to outliers (Jobson and Korkie 1980). In sharp contrast to the shrinkage estimator of Ledoit and Wolf (2004), the GS does not rely on the sample covariance matrix as input.

The remainder of this article is organized as follows. We first provide an introduction to the GS. We then introduce the dataset and the backtesting framework employed to compare the empirical performance of HC, shrinkage estimation, and the GS. The core empirical analysis is then presented. Operating within Markowitz's mean–variance portfolio optimization framework, we find that the GS's returns, in almost all investment scenarios considered, dominate those from both HC and from shrinkage estimation. The Appendix contains the necessary technical aspects for this article, which follow from the next section.

THE GERBER STATISTIC

The purpose of this section (and its continuation in the Appendix) is to introduce the GS and the corresponding Gerber correlation matrix, which is then converted to a

[1] The first preprint on the Gerber statistic (Gerber, Markowitz, and Pujara 2015) was posted on SSRN in 2015, but it was removed owing to some errors in calculation. It is currently available via www.stat.rice.edu/~pe6/Gerber2015.pdf. The present article constitutes the final draft of this preprint. All figures and results in this work are fully reproducible via the resources provided on the GitHub page https://github.com/yinsenm/gerber.

Gerber covariance matrix that is inputted into the mean–variance portfolio optimizer. We begin with the necessary notation for the GS's formulation.

Notation

We consider $k = 1, \ldots, K$ securities and $t = 1, \ldots, T$ time periods. Let r_{tk} be the return of security k at time t. For each pair (i, j) of assets for each time t, we convert the return observation of pair (r_{ti}, r_{tj}) to a joint observation $m_{ij}(t)$ given by the assignment mechanism:

$$m_{ij}(t) = \begin{cases} +1 \text{ if } r_{ti} \geq +H_i \text{ and } r_{tj} \geq +H_j, \\ +1 \text{ if } r_{ti} \leq -H_i \text{ and } r_{tj} \leq -H_j, \\ -1 \text{ if } r_{ti} \geq +H_i \text{ and } r_{tj} \leq -H_j, \\ -1 \text{ if } r_{ti} \leq -H_i \text{ and } r_{tj} \geq +H_j, \\ 0 \text{ otherwise,} \end{cases} \tag{1}$$

where H_k is a threshold for security k and is calculated as

$$H_K = cs_k, \tag{2}$$

where c is some fraction (typically set to 1/2, but may also be increased to 7/10 or 9/10) and s_k is the sample standard deviation of the return of security k.[2] There are three key takeaways from the display in Equation 1:

- The joint observation $m_{ij}(t)$ is set to +1 if the series i and j simultaneously pierce their thresholds in the same direction at time t.
- The joint observation $m_{ij}(t)$ is set to –1 if the series i and j simultaneously pierce their thresholds in opposite directions at time t.
- The joint observation $m_{ij}(t)$ is set to 0 if at least one of the series does not pierce its threshold at time t.

We now consider the following statistic for a pair of assets

$$g_{ij} = \frac{\sum_{t=1}^{T} m_{ij}(t)}{\sum_{t=1}^{T} \left| m_{ij}(t) \right|}. \tag{3}$$

Because the statistic in Equation 3 relies on counts of the number of simultaneous piercings of thresholds, and not on the extent to which the thresholds are pierced, it is insensitive to extreme movements that distort product-moment–based measures. At the same time, because a series must exceed its threshold before it becomes a candidate to be counted (i.e., it is given a value of m_{ij} that is either +1 or –1), the statistic in Equation 3 is also insensitive to small movements that may simply be noise.

A fundamental tenet of modern portfolio theory is that the covariance matrix of securities' returns must be positive semidefinite. However, when working with real data, we found that the covariance matrix corresponding to the statistic in Equation 3 was often not positive semidefinite. This led us to develop an alternative form of the

[2] More robust measures than standard deviation could, of course, be used for the threshold computation, but this is beyond the scope of the present work.

statistic in Equation 3 that gives rise to a positive semidefinite covariance matrix. We call this alternative form, appearing in Equation A-3 in the Appendix, the GS. The GS closely resembles the statistic displayed in Equation 3. The Appendix contains a more detailed technical discussion of the GS's formulation.

In concordance with Pearson's correlation coefficient (and Kendall's Tau coefficient), the value of the GS is also always contained in the interval [–1, 1]. However, there are key conceptual differences between the GS and the Pearson correlation coefficient:

- The Pearson correlation coefficient inputs the sample covariance of assets *i* and *j* and the sample standard deviation of assets *i* and *j* (and therefore the sample means of assets *i* and *j*). By definition, the sample covariance, the sample mean, and the sample standard deviation are calculated over *all* data points, regardless of whether the points correspond to meaningful co-movement or to pure noise. This causes the Pearson correlation to be highly sensitive to small co-movements that may be due to noise alone. In contrast, the numerator of the GS in Equation A-3 only includes the *subset* of the dataset containing the points corresponding to meaningful co-movement; that is, the GS strips away noisy data. We see this to be the key reason why the GS is a more robust co-movement measure than the standard Pearson correlation.
- Unlike the Pearson correlation coefficient, the GS's formulation need not require any estimates of moments. Indeed, we could achieve an entirely moment-free framework for the GS by replacing s_k in Equation 2 with a more robust measure of standard deviation (as previously noted in footnote 2). We shall explore candidates for this measure in future work.

EMPIRICAL STUDY

In this section, we commence the empirical study of the performance of the GS in comparison to two commonly used methods of covariance estimation for the purpose of portfolio construction: HC and the shrinkage estimator of Ledoit and Wolf (2004).

Dataset

The dataset we consider is a well-diversified collection of nine assets over the time period January 1988 to December 2020:

1. S&P 500 index (US large-cap stocks; ticker SPX)
2. Russell 2000 index (US small-cap stocks; ticker RTY)
3. MSCI EAFE index (captures large- and mid-cap equities across 21 developed countries excluding the United States and Canada; ticker MXEA)
4. MSCI Emerging Markets index (captures large- and mid-cap equities across 27 emerging markets; ticker MXEF)
5. Bloomberg Barclays US Aggregate Bond index (includes Treasuries and government-related and corporate securities; ticker LBUSTRUU)
6. Bloomberg Barclays US Corporate High Yield Bond index (ticker LF98TRUU)
7. Real estate FTSE NAREIT all equity REITS index (ticker FNERTR)
8. Gold (ticker XAU)
9. S&P GSCI Goldman Sachs Commodity index (ticker SPGSCI)

EXHIBIT 1

Descriptive Statistics for the Nine-Asset Portfolio

Index	Arithmetic Return (%)	Geometric Return (%)	Annualized SD (%)
S&P 500 TR	11.90	10.47	14.63
Russel 2000 TR	11.74	10.14	19.36
MSCI EAFE TR	6.60	4.77	16.89
MSCI Emerging Market TR	12.13	7.61	22.31
US Agg Bond TR	6.11	6.00	3.57
US Corp High-Yield Bond TR	9.35	8.36	8.74
REITS TR	11.97	10.37	18.32
Gold TR	6.04	5.03	15.26
S&P GSCI TR	5.27	2.36	21.48

NOTES: The descriptive statistics are computed using monthly data from January 1990 to December 2020. TR denotes the total return data, which accounts for asset appreciation from both capital gains and reinvesting of dividends.

The monthly total returns (TR) for these nine assets for the period from January 1988 to December 2020 were obtained from a Bloomberg terminal. Each asset contains 396 observations over this time period. The TR indexes provided by Bloomberg track capital gains and account for cash distributions such as dividends or interest through asset reinvesting. As we shall detail in the sequel, our backtesting procedure for MVO requires two years of monthly returns to initialize the first portfolio. The descriptive statistics for the monthly total returns of the nine-asset portfolio in Exhibit 1 are therefore calculated from the period January 1990 to December 2020 rather than from the period January 1988 to December 2020.

Because the nine assets are well diversified, we do not expect to observe a strong pairwise correlation structure between the assets. This is confirmed by Exhibit 2, which displays a correlation matrix of the total return series from January 1990 to December 2020 for the nine assets.

Competing Methods

As previously discussed, the two competing methods to the GS we shall consider are the HC matrix and the SM of Ledoit and Wolf (2004). The HC matrix is computed from the sample correlation matrix calculated via the standard Pearson correlation (Jobson and Korkie 1980). The shrinkage estimator introduced by Ledoit and Wolf (2004) is a convex combination of a structure covariance matrix and the sample HC matrix; the sample covariance matrix is shrunk toward a targeted structured estimator.

We proceed to highlight some critical conceptual differences between the GS and the SM of Ledoit and Wolf (2004):

1. The SM directly inputs the sample covariance matrix. In contrast, the GS does *not* rely upon the sample covariance matrix as input. In lieu, the GS computes concordant and discordant pair counts. The GS thereby offers a natural extension of Kendall's Tau for use in portfolio management.
2. The GS's framework for considering concordant and discordant pairs of assets in one dataset (in the present article, the dataset of historical returns) can be naturally extended to working with multiple datasets. For example, suppose a portfolio manager wished to consider three datasets simultaneously: historical returns, trading volume, and implied volatility. Furthermore, suppose that this portfolio manager wished to deem two assets *A* and *B* to be concordant if the return for both assets is higher than *x*% at the same time that the trading volume increases by more than *y*% and the implied volatility increases by more than *z*%, where *x*, *y*, and *z* are any (finite) real numbers greater than zero. The GS provides a natural framework for designing this rule (as well as more sophisticated rule-based systems) and computing the corresponding GSs. In contrast, moment-based methods such as shrinkage and HC do not provide the same natural framework for considering such rule-based systems for determining co-movement.

As for similarities between the GS and the SM of Ledoit and Wolf (2004), it should be noted that the Ledoit and Wolf shrinkage constant controls the degree to which

EXHIBIT 2

Heat Map of the Correlation Matrix (given the total return series from January 1990 to December 2020) for the Nine Assets

	RTY	MXEA	MXEF	XAU	SPGSCI	LF98TRUU	LBUSTRUU	FNERTR
SPX	0.82	0.75	0.67	−0.03	0.25	0.62	0.12	0.57
RTY		0.66	0.67	0.01	0.29	0.64	0	0.64
MXEA			0.71	0.11	0.33	0.57	0.08	0.5
MXEF				0.19	0.32	0.59	0.02	0.46
XAU					0.26	0.09	0.19	0.08
SPGSCI						0.28	−0.06	0.16
LF98TRUU							0.22	0.61
LBUSTRUU								0.21

the sample covariance matrix is shrunk. The analog for the GS is the value *c* given in Equation 2; we recall that the value of *c* determines the magnitude of the threshold H_k.

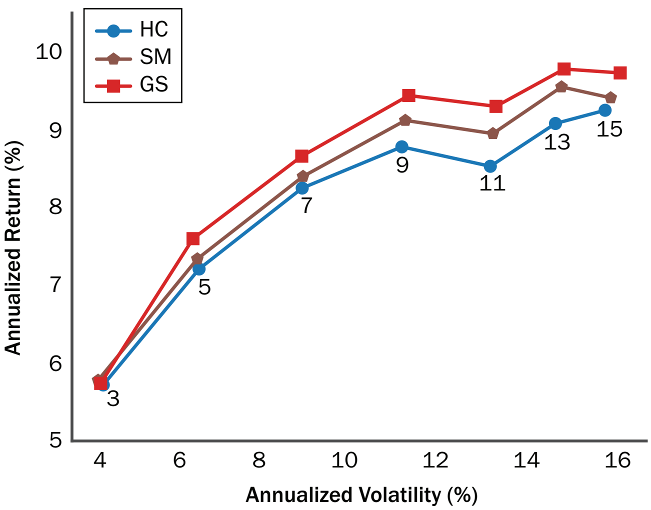
Optimization Procedure and Portfolio Backtesting

The portfolio optimization framework we shall consider is that of MVO (Markowitz 1952, 1959) with a transaction fee of 10 bps, or 0.1% of traded volume. The following backtesting procedure is employed to benchmark performance among different covariance estimators for portfolio optimization.

- Beginning January 1990, at the beginning of each month, the monthly returns of the current list of assets from a lookback window of 24 months are used to estimate the expected return vector and the covariance matrix, each of which is then inputted into the Markowitz MVO procedure.
- All portfolios are rebalanced on a monthly basis. We repeat this rebalancing process by moving the in-sample period one month forward and computing the updated efficient portfolio for the next month. This rolling-window investing procedure offers the advantage of being more adaptive to market structural changes and helps to ameliorate data mining bias. Because two years' worth of monthly returns are required to initialize the first portfolio, our performance evaluation ranges from January 1990 to December 2020.

EXHIBIT 3

Realized Performance in Terms of Annualized Return and Annualized Volatility of Portfolios (ex post efficient frontiers) with Different Risk Target Levels from 3% to 15%, with an Increment of 2%, given the Gerber Threshold $c = 0.5$



NOTE: The blue frontier illustrates the ex post performance of HC-based portfolios, the brown frontier presents the ex post performance of SM-based portfolios, and the red frontier corresponds to the ex post performance of GS-based portfolios.

EMPIRICAL RESULTS

We now present the article's key empirical results. Working with the dataset introduced in the "Dataset" section and employing the transaction costs and backtesting algorithm given in the previous section, we consider the performance of the GS in comparison to HC and to the SM of Ledoit and Wolf (2004) for three different values of the Gerber threshold c given in Equation 2: $c = 0.5$, $c = 0.7$, and $c = 0.9$.

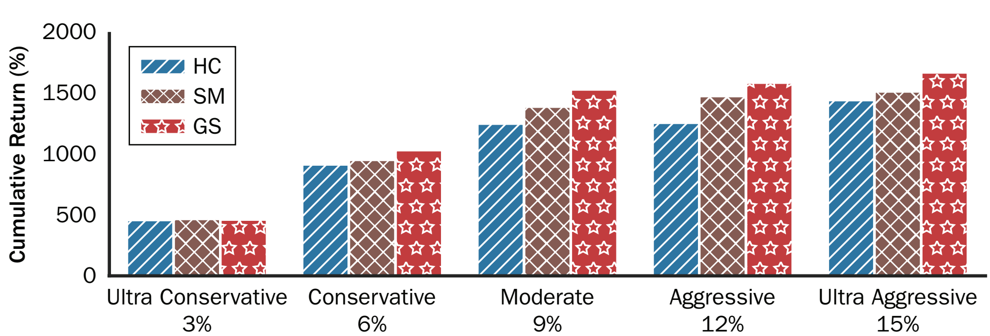
Gerber Statistic with $c = 0.5$

We first study the GS with a threshold value of $c = 0.5$. We report four key findings:

1. For all risk target levels, the GS offers a more favorable risk–return profile than HC. With the exception of the ultra-conservative risk target level of 3%, the GS offers a more favorable risk–return profile than SM (Exhibit 3).
2. For all risk targets, the GS yields higher cumulative returns than HC. With the exception of the very conservative risk target level of 3%, the GS yields higher cumulative returns than SM (Exhibits 4 and 5).
3. With values of portfolio turnover, skewness, and kurtosis similar to both the HC and SM portfolios, the GS posts higher geometric returns and higher Sharpe ratios than HC across all risk target levels (Exhibit 6). With the exception of the very conservative risk target level of 3% (Exhibit 6), the GS yields higher geometric returns and higher Sharpe ratios than SM across all other risk target levels.

EXHIBIT 4

Cumulative Returns in Percentage (from 1990 to 2020) for HC-Based Portfolios, SM-Based Portfolios, and GS-Based Portfolios at Five Different Annual Risk Target Levels (3%, 6%, 9%, 12%, and 15%), given the Gerber Threshold $c = 0.5$



NOTE: The calculation assumes that $100,000 is invested in January 1990 and is left to grow according to portfolio weights determined by each covariance method and risk target level until December 2020.

EXHIBIT 5

Account Dollar Value in December 2020 for HC-Based Portfolios, SM-Based Portfolios, and GS-Based Portfolios at Five Different Annual Risk Target Levels (3%, 6%, 9%, 12%, and 15%), given the Gerber Threshold $c = 0.5$

Method	HC	SM	GS
Ultra-Conservative (3%)	$561,276.27	$570,161.82	$564,972.97
Conservative (6%)	$1,020,099.74	$1,058,096.16	$1,138,042.25
Moderate (9%)	$1,356,911.18	$1,497,204.43	$1,639,089.77
Aggressive (12%)	$1,364,148.39	$1,584,491.30	$1,695,255.50
Ultra-Aggressive (15%)	$1,551,338.93	$1,622,590.70	$1,779,756.04

NOTE: The calculation assumes that $100,000 is invested in January 1990 and is left to grow according to portfolio weights determined by each covariance method and risk target level until December 2020.

4. For some risk target levels, the average annualized geometric return of GS is more than 30 bps higher than that of SM and more than 75 bps higher than that of HC. The latter result is unsurprising given the limitations of HC (Jobson and Korkie 1980; Ledoit and Wolf 2004), and so we instead focus on the advantages of GS over SM. For the 9% risk target level, the average annualized geometric return of GS is approximately 32 bps higher than that of SM, and its cumulative return is 10.16% higher than that for SM over the 1990–2020 period (Exhibit 6). For the 15% risk target level, the average annualized geometric return of GS is approximately 32 bps higher than that of SM, and its cumulative return is 10.32% higher than that of SM over the 1990–2020 period (Exhibit 6).

Gerber Statistic with $c = 0.7$

We proceed to study the GS with a value of $c = 0.7$. We report four key findings:

1. For all risk target levels, the GS offers a more favorable risk–return profile than both HC and SM (Exhibit 7).
2. For all risk target levels, the GS offers superior cumulative returns to both HC and SM (Exhibits 8 and 9).
3. For all risk target levels, the GS gives higher geometric returns and Sharpe ratios compared to both HC and SM, and it has similar values of portfolio turnover, skewness, and kurtosis to HC and SM (Exhibits 6 and 10).
4. For some risk target levels, the average annualized geometric return of GS is more than 40 bps higher than that of SM and is more than 90 bps higher than that of HC. The latter finding is unsurprising given the limitations of HC (Jobson and Korkie 1980; Ledoit and Wolf 2004), and so we instead focus on the advantages of GS over SM. For the 12% risk target level, the average annualized geometric return of GS is approximately 41 bps higher than that of SM, and its cumulative return is 13.12% higher than that of SM over the 1990–2020 period (Exhibits 6 and 10). For the 15% risk target level, the average annualized geometric return of GS is approximately 35 bps higher than that of SM, and its cumulative return is 11.18% higher than that of SM over the 1990–2020 period (Exhibits 6 and 10).

EXHIBIT 6

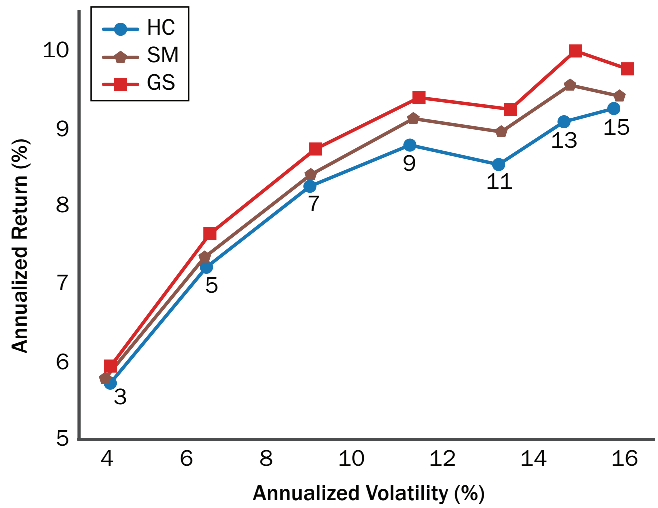
Performance Metrics for HC-, SM-, and GS-Based Portfolios at Five Different Risk Target Levels (3%, 6%, 9%, 12%, and 15%) for the Full Testing Period between January 1990 and December 2020

	Ultra-Conservative (3%)			Conservative (6%)			Moderate (9%)			Aggressive (12%)			Ultra-Aggressive (15%)		
Covariance Method	**HC**	**SM**	**GS**	**HC**	**SM**	**GS**	**HC**	**SM**	**GS**	**HC**	**SM**	**GS**	**HC**	**SM**	**GS**
Arithmetic Return (%)	5.83	5.90	5.85	8.10	8.22	8.46	9.41	9.77	10.09	9.69	10.32	10.50	10.54	10.78	11.17
Geometric Return (%)	5.72	5.78	5.74	7.78	7.91	8.16	8.78	9.12	9.44	8.79	9.32	9.56	9.25	9.41	9.73
Cumulative Return (%)	461.28	470.16	464.97	920.10	958.10	1,038.04	1,256.91	1,397.20	1,539.09	1,264.15	1,484.49	1,595.26	1,451.34	1,522.59	1,679.76
Annualized SD (%)	4.07	3.95	4.01	7.49	7.50	7.43	10.99	11.07	11.15	13.82	13.92	13.98	15.70	15.83	16.05
Annualized Skewness	–0.90	–0.90	–0.88	–0.93	–0.98	–0.99	–0.85	–0.88	–0.93	–0.93	–0.92	–0.93	–0.86	–0.86	–0.87
Annualized Kurtosis	5.21	5.37	5.31	5.53	5.85	6.07	4.97	5.11	5.60	5.27	5.30	5.56	5.34	5.42	5.55
Maximum Drawdown (%)	–10.03	–7.67	–7.73	–20.30	–19.10	–17.74	–28.83	–28.03	–26.58	–36.47	–35.22	–32.47	–41.76	–41.13	–43.01
Monthly 95% VaR (%)	–1.53	–1.52	–1.58	–2.85	–2.69	–2.68	–4.35	–4.29	–4.29	–5.99	–5.81	–5.84	–6.70	–6.59	–6.42
Sharpe Ratio	0.73	0.77	0.75	0.67	0.69	0.73	0.55	0.58	0.60	0.44	0.47	0.49	0.42	0.42	0.44
Annualized Turnover	1.80	1.39	1.58	2.78	2.53	2.49	3.69	3.40	3.37	4.45	4.27	4.19	4.46	4.33	4.23

NOTES: The three-month US T-bill rate was used as the risk-free rate. The transaction cost is modeled as 10 bps of the traded volume for each rebalancing event.

EXHIBIT 7

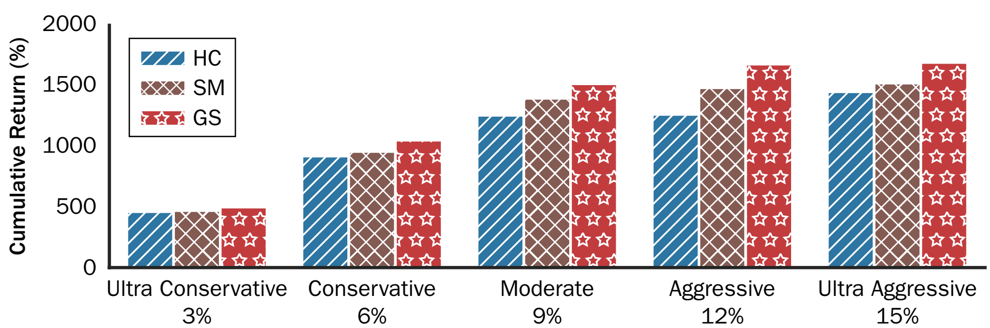
Realized Performance in Terms of Annualized Return and Annualized Volatility of Portfolios (ex post efficient frontiers) with Different Risk Target Levels from 3% to 15%, with an Increment of 2%, given the Gerber Threshold $c = 0.7$



NOTE: The blue frontier illustrates the ex post performance of HC-based portfolios, the brown frontier presents the ex post performance of SM-based portfolios, and the red frontier corresponds to the ex post performance of the GS-based portfolios.

EXHIBIT 8

Cumulative Returns in Percentage (from 1990 to 2020) for HC-Based Portfolios, SM-Based Portfolios, and GS-Based Portfolios at Five Different Annual Risk Target Levels (3%, 6%, 9%, 12%, and 15%), given the Gerber Threshold $c = 0.7$



NOTE: The calculation assumes that $100,000 is invested in January 1990 and is left to grow according to portfolio weights determined by each covariance method and risk target level until December 2020.

EXHIBIT 9

Dollar Value in December 2020 for HC-Based Portfolios, SM-Based Portfolios, and GS-Based Portfolios under Five Annual Risk Target Levels (3%, 6%, 9%, 12%, and 15%), with a Gerber Threshold $c = 0.7$

Method	HC	SM	GS
Ultra-Conservative (3%)	$561,276.27	$570,161.82	$599,043.19
Conservative (6%)	$1,020,099.74	$1,058,096.16	$1,151,575.55
Moderate (9%)	$1,356,911.18	$1,497,204.43	$1,617,126.27
Aggressive (12%)	$1,364,148.39	$1,584,491.30	$1,779,267.42
Ultra-Aggressive (15%)	$1,551,338.93	$1,622,590.70	$1,792,846.87

NOTE: The calculation assumes that $100,000 is invested in January 1990 and is left to grow until December 2020.

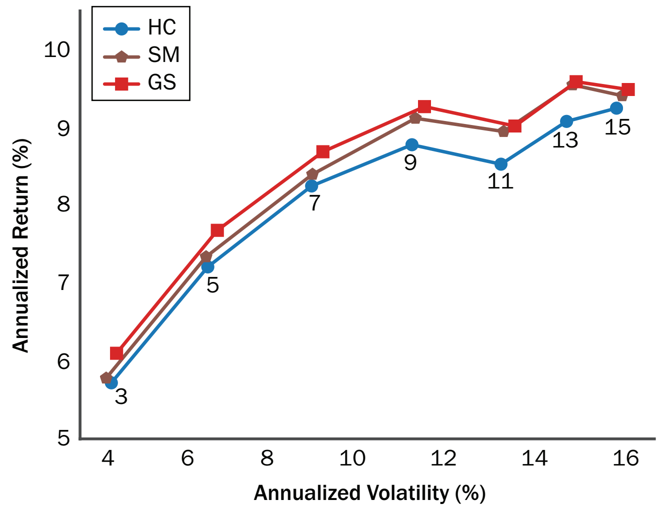
EXHIBIT 10

Performance Sensitivity Study of the GS-Based Portfolios for Thresholds c = 0.5, c = 0.7, and c = 0.9 at Five Different Risk Target Levels (3%, 6%, 9%, 12%, and 15%)

	Ultra-Conservative (3%)			Conservative (6%)			Moderate (9%)			Aggressive (12%)			Ultra-Aggressive (15%)		
GS Threshold c	0.50	0.70	0.90	0.50	0.70	0.90	0.50	0.70	0.90	0.50	0.70	0.90	0.50	0.70	0.90
Arithmetic Return (%)	5.85	6.06	6.22	8.46	8.52	8.59	10.09	10.05	9.94	10.50	10.68	10.34	11.17	11.17	10.90
Geometric Return (%)	5.74	5.94	6.10	8.16	8.20	8.26	9.44	9.39	9.27	9.56	9.73	9.37	9.73	9.76	9.49
Cumulative Return (%)	464.97	499.04	527.08	1,038.04	1,051.58	1,069.91	1,539.09	1,517.13	1,460.20	1,595.26	1,679.27	1,507.57	1,679.76	1,692.85	1,560.63
Annualized SD (%)	4.01	4.08	4.19	7.43	7.62	7.76	11.15	11.20	11.28	13.98	14.02	14.15	16.05	16.01	15.97
Annualized Skewness	–0.88	–1.06	–1.04	–0.99	–1.04	–0.98	–0.93	–0.90	–0.87	–0.93	–0.94	–0.98	–0.87	–0.89	–0.86
Annualized Kurtosis	5.31	6.26	6.13	6.07	6.30	5.84	5.60	5.42	5.12	5.56	5.58	5.66	5.55	5.59	5.38
Maximum Drawdown (%)	–7.73	–9.11	–9.67	–17.74	–19.08	–20.03	–26.58	–27.75	–28.12	–32.47	–33.94	–34.01	–43.01	–43.12	–40.94
Monthly 95% VaR (%)	–1.58	–1.50	–1.55	–2.68	–2.79	–2.82	–4.29	–4.37	–4.40	–5.84	–5.78	–5.71	–6.42	–6.49	–6.53
Sharpe Ratio	0.75	0.78	0.80	0.73	0.72	0.71	0.60	0.60	0.58	0.49	0.50	0.47	0.44	0.44	0.42
Annualized Turnover	1.58	1.49	1.51	2.49	2.55	2.58	3.37	3.43	3.44	4.19	4.19	4.26	4.23	4.21	4.32

EXHIBIT 11

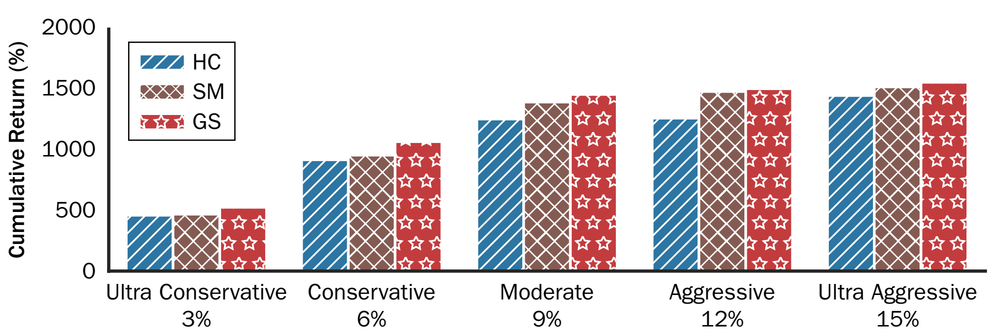
Realized Performance in Terms of Annualized Return and Annualized Volatility of Portfolios (ex post efficient frontiers) with Different Risk Target Levels from 3% to 15%, with an Increment of 2%, given the Gerber Threshold $c = 0.9$



NOTE: The blue frontier illustrates the ex post performance of HC-based portfolios, the brown frontier presents the ex post performance of SM-based portfolios, and the red frontier corresponds to the ex post performance of the GS-based portfolios.

EXHIBIT 12

Cumulative Returns in Percentage (from 1990 to 2020) for HC-Based Portfolios, SM-Based Portfolios, and GS-Based Portfolios at Five Different Annual Risk Target Levels (3%, 6%, 9%, 12%, and 15%), given the Gerber Threshold $c = 0.9$



NOTE: The calculation assumes that $100,000 is invested in January 1990 and is left to grow according to portfolio weights determined by each covariance method and risk target level until December 2020.

Gerber Statistic with $c = 0.9$

We proceed to study the GS with a value of $c = 0.9$. We report four key findings:

1. For all risk target levels, the GS offers a more favorable risk–return profile than both HC and SM (Exhibit 11).
2. For all risk target levels, the GS offers superior cumulative returns to both HC and SM (Exhibits 12 and 13).
3. For all risk target levels, the GS posts higher geometric returns and Sharpe ratios compared to both SM and HC, and it has similar values of portfolio turnover, skewness, and kurtosis (Exhibits 6 and 10).

EXHIBIT 13

Account Dollar Value in December 2020 for HC-Based Portfolios, SM-Based Portfolios, and GS-Based Portfolios at Five Different Annual Risk Target Levels (3%, 6%, 9%, 12%, and 15%), given the Gerber Threshold $c = 0.9$

Method	HC	SM	GS
Ultra-Conservative (3%)	$561,276.27	$570,161.82	$627,077.52
Conservative (6%)	$1,020,099.74	$1,058,096.16	$1,169,911.24
Moderate (9%)	$1,356,911.18	$1,497,204.43	$1,560,198.02
Aggressive (12%)	$1,364,148.39	$1,584,491.30	$1,607,572.21
Ultra-Aggressive (15%)	$1,551,338.93	$1,622,590.70	$1,660,628.94

NOTE: The calculation assumes that $100,000 is invested in January 1990 and is left to grow according to portfolio weights determined by each covariance method and risk target level until December 2020.

4. For the 3% and 6% risk target levels, the average annualized geometric return of GS is, respectively, approximately 32 and 35 bps higher than those of SM. The cumulative returns are, respectively, 12.11% and 11.67% higher than those of SM over the 1990–2020 period (Exhibits 6 and 10). We also note that for the 6% risk target level, the average annualized geometric return of GS is more than 48 bps higher than that of HC.

CONCLUSION

This article has introduced a co-movement measure called the GS. The GS is well suited for assessing co-movement between financial time series because it is insensitive to extremely large co-movements that distort product-moment–based measures and to small movements that are likely to be noise. We have studied the performance of the GS within the mean–variance portfolio optimization framework of Markowitz (1952, 1959). In every investment scenario considered, the GS's performance is superior to that of HC. In almost every[3] investment scenario considered, the GS dominates the shrinkage estimator on the key metrics of interest to any investor: cumulative return, average geometric return, and Sharpe ratio. An additional advantage of the GS lies in the fact that, unlike the shrinkage method, it does not rely upon the sample covariance matrix as input. Finally, the GS is easy to compute and is straightforward to implement in any MVO software. Our hope is that it will become a welcome alternative to both HC and to the shrinkage estimator of Ledoit and Wolf (2004).

APPENDIX

The first subsection of this Appendix documents the formulation of the Gerber statistic in Equation A-3. The second subsection provides a simple example to illustrate how the Gerber statistic is calculated.

[3] We have considered five risk targets for Gerber thresholds c = 0.5, c = 0.7, and c = 0.9. This amounts to 15 investment scenarios in total. The shrinkage estimator of Ledoit and Wolf (2004) only dominates the Gerber statistic in one of these 15 investment scenarios, corresponding to c = 0.5 and a very conservative risk target level of 3%.

EXHIBIT A1

Graphical Relationship Between Two Securities

UD	*UN*	*UU*
ND	*NN*	*NU*
DD	*DN*	*DU*

FORMULATION OF GERBER STATISTIC IN EQUATION A-3

Recall that in Equation 3 we defined a statistic for a pair of assets as follows:

$$g_{ij} = \frac{\sum_{t=1}^{T} m_{ij}(t)}{\sum_{t=1}^{T} \left| m_{ij}(t) \right|}.$$

Let us refer to a pair for which both components pierce their thresholds while moving in the same direction as a concordant pair, and to one whose components pierce their thresholds while moving in opposite directions as a discordant pair. Letting n_{ij}^{c} be the number of concordant pairs for series i and j, and letting n_{ij}^{d} be the number of discordant pairs, Equation 3 is immediately equivalent to

$$g_{ij} = \frac{n_{ij}^{c} - n_{ij}^{d}}{n_{ij}^{c} + n_{ij}^{d}}. \tag{A-1}$$

Note that the statistic in Equation A-1 is identical to Kendall's Tau if the threshold H_k is set to zero for all k.

We now consider the following graphical representation for the relationship between two securities in Exhibit A1. U represents the case in which a security's return lies above the upper threshold (i.e., is up), N represents the case in which a security's return lies between the upper and lower thresholds (i.e., is neutral), and D represents the case in which a security's return lies below the lower threshold (i.e., is down). In Exhibit A1, the rows represent categorizations of security i and the columns represent categorizations of security j. The boundaries between the rows and the columns are the chosen thresholds. For example, if at time t the return of security i is above the upper threshold, this observation lies in the top row. If, at the same time t, the return of security j lies between the two thresholds, this observation lies in the middle column. Therefore, this specific observation would lie in the *UN* region.

Over the history $t = 1, \ldots, T$, there will be observations scattered over the nine regions. Let n_{ij}^{pq} be the number of observations for which the returns of securities i and j lie in regions p and q, respectively, for $p, q \in \{U, N, D\}$. With this notation in hand, the following is an equivalent expression to the statistic presented in Equation A-1:

$$g_{ij} = \frac{n_{ij}^{UU} + n_{ij}^{DD} - n_{ij}^{UD} - n_{ij}^{DU}}{n_{ij}^{UU} + n_{ij}^{DD} + n_{ij}^{UD} + n_{ij}^{DU}}. \tag{A-2}$$

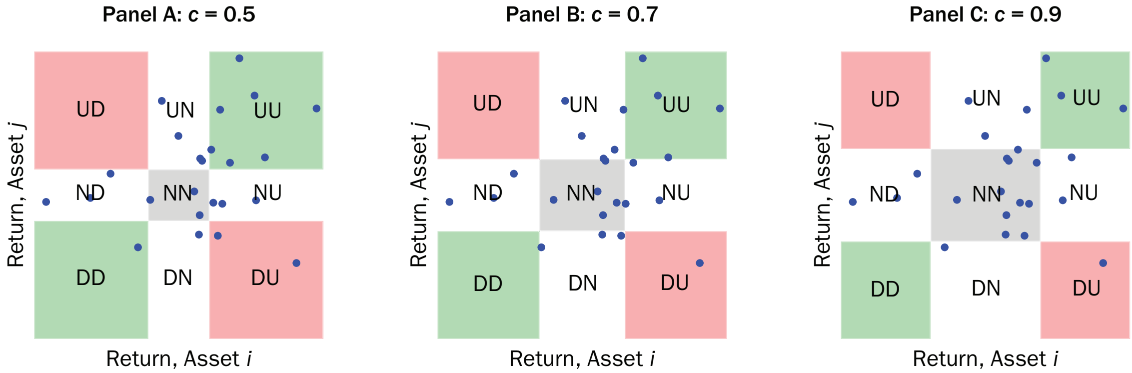
As previously noted, we must alter the denominator in Equation A-1 to obtain a Gerber matrix that yields a corresponding covariance matrix in positive semidefinite form. Our alternative choice, which we call the *Gerber statistic*, is

$$g_{ij} = \frac{n_{ij}^{UU} + n_{ij}^{DD} - n_{ij}^{UD} - n_{ij}^{DU}}{T - n_{ij}^{NN}}. \tag{A-3}$$

The Gerber matrix $\boldsymbol{G}$ is the matrix that contains the Gerber statistic g_{ij} in its ith row and jth column. In the empirical studies performed, and for all cases of Gerber thresholds c considered, we always observe the covariance matrix obtained from the Gerber matrix $\boldsymbol{G}$ to be positive semidefinite.

EXHIBIT A2

Illustration of Pairwise Returns for Evaluating the Gerber Statistics given c = 0.5, c = 0.7, and c = 0.9



NOTES: Each pairwise monthly return appears as a blue dot. The points in the green zone correspond to the concordant pairs, whereas the points falling in the red zone are discordant pairs. The return series of assets *i* and *j* are transformed to {–1, 0, 1} using upper and lower thresholds calculated by Equation 2.

CALCULATION OF THE GERBER STATISTIC

We provide a brief example to illustrate how the Gerber statistic is calculated between a given pair of assets. In Exhibit A2, we compute 24 pairwise monthly returns between the assets S&P 500 (SPX) and Gold (XAU) for the period from January 2019 to December 2020. Recalling that the threshold of H_k as defined in Equation 2 may be altered, we consider three different values of c: $c = 0.5$, $c = 0.7$, and $c = 0.9$.

The key intuition for our choice of the Gerber statistic's denominator in Equation A-3 comes from the following observation: as c becomes larger, more data points are included in the region *NN*. This leads to the statistic becoming more robust and less sensitive to noise in the data. We refer to this artifact of the Gerber statistic as stripping noise from the data.

We now calculate the Gerber statistic by counting the points falling into each region. The results for the three cases $c = 0.5$, $c = 0.7$, and $c = 0.9$ are given below. All result in Gerber statistics that differ from the standard Pearson correlation coefficient of 0.22.

1. In the case $c = 0.5$ (Exhibit A2, Panel A), the counts for nine regions are $n_{ij}^{UD} = 0$, $n_{ij}^{UN} = 4$, $n_{ij}^{UU} = 7$, $n_{ij}^{ND} = 3$, $n_{ij}^{NN} = 3$, $n_{ij}^{NU} = 3$, $n_{ij}^{DD} = 1$, $n_{ij}^{DN} = 1$, and $n_{ij}^{DU} = 2$. Employing the formula or the Gerber statistic in Equation A-3, we find that

$$g_{ij} = \frac{7+1-0-2}{24-3} = \frac{2}{7} \approx 0.286.$$

2. In the case $c = 0.7$ (Exhibit A2, Panel B), the counts for nine regions are $n_{ij}^{UD} = 0$, $n_{ij}^{UN} = 5$, $n_{ij}^{UU} = 4$, $n_{ij}^{ND} = 3$, $n_{ij}^{NN} = 6$, $n_{ij}^{NU} = 2$, $n_{ij}^{DD} = 0$, $n_{ij}^{DN} = 3$, and $n_{ij}^{DU} = 1$. Employing the formula for the Gerber statistic in Equation A-3, we find that

$$g_{ij} = \frac{4+0-0-1}{24-6} = \frac{1}{6} \approx 0.166.$$

3. In the case $c = 0.9$ (Exhibit A2, Panel C), the counts for nine regions are $n_{ij}^{UD} = 0$, $n_{ij}^{UN} = 3$, $n_{ij}^{UU} = 3$, $n_{ij}^{ND} = 3$, $n_{ij}^{NN} = 11$, $n_{ij}^{NU} = 2$, $n_{ij}^{DD} = 0$, $n_{ij}^{DN} = 1$, and $n_{ij}^{DU} = 1$. Employing the formula for the Gerber statistic in Equation A-3, we find that

$$g_{ij} = \frac{3+0-0-1}{24-11} = \frac{2}{13} \approx 0.154.$$

ACKNOWLEDGMENTS

We wish to thank David Starer and Jacques Friedman for helpful conversations.

REFERENCES

Gerber, S., H. M. Markowitz, and P. Pujara. "Enhancing Multi-Asset Portfolio Construction under Modern Portfolio Theory with a Robust Co-Movement Measure." *Social Science Research Network Working Paper Series*, 2015.

Hampel, F. R. "Contributions to the Theory of Robust Estimation." PhD thesis, University of California, Berkeley, 1968.

——. 1974. "The Influence Curve and Its Role in Robust Estimation." *Journal of the American Statistical Association* 69 (346): 383–393.

Huber, P. J. "Robust Statistical Procedures." Philadelphia: Society for Industrial and Applied Mathematics, 1977.

Jobson, J. D., and B. Korkie. 1980. "Estimation for Markowitz Efficient Portfolios." *Journal of the American Statistical Association* 75 (371): 544–554.

Kendall, M. G. 1938. "A New Measure of Rank Correlation." *Biometrika* 30 (1/2): 81–93.

Ledoit, O., and M. Wolf. 2004. "Honey, I Shrunk the Sample Covariance Matrix." *The Journal of Portfolio Management* 30 (4): 110–119.

Markowitz, H. M. 1952. "Portfolio Selection." *The Journal of Finance* 7 (1): 77–91.

——. "Portfolio Selection: Efficient Diversification of Investments." Cambridge, MA: Basil Blackwell, 1959.

Sharpe, W. F. 1963. "A Simplified Model for Portfolio Analysis." *Management Science* 9 (2): 277–293.

Shevlyakov, G., and P. Smirnov. 2011. "Robust Estimation of the Correlation Coefficient: An Attempt of Survey." *Austrian Journal of Statistics* 40 (1&2): 147–156.

Tukey, J. W. "Contributions to Probability and Statistics: Essays in Honor of Harold Hotelling." In A Survey of Sampling from Contaminated Distributions. Stanford: Stanford University Press, 1960.